ADDENDUM to THEOREM 10.4 in "BOUNDARIES OF ANALYTIC VARIETIES"

by

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The main result of [HL], put simply, provides a characterization of the boundaries of complex subvarieties in C^n. One of the minor applications of this result, namely Theorem 10.4, requires clarification because of the note [LY] of Luk-Yau.(See [E].) The intent of [LY] is to provide a counterexample to the boundary regularity assertion of Theorem 10.4. However, Theorem 10.4 is fundamentally correct. Furthermore, the authors of [LY] seem not to have realized that their example already appears in [HL] (Example 9.1). This example is simply an immersion $C^2 \to C^3$ which folds back on itself, and therefore when restricted to balls gives rise to crossing singularities both in the interior and at the boundary. It shows that for boundaries M which are embedded and strongly pseudoconvex, the fill-in variety may not be embedded.

The boundary regularity stated in Theorem 10.4 conforms to this example. The possibility of crossing singularities is explicitly stated in the last line of the theorem where multiple local components of V at the boundary are discussed.

There was a minor error in the exposition of Theorem 10.4. Since this mis-statement was internally contradictory, the correct version may have been evident to the reader. Nevertheless, in this note we amend the error. We also give an alternative version of the result which we thought was obvious, but perhaps was not. In addition this note corrects a result of Stephen Yau [Y], demonstrates that Lempert's use of Theorem 10.4 carries through, and shows how the unproven theorem in [LY] follows trivially from our paper.

Incidentally, Theorem 10.4 was not a new result for $\dim(V) \geq 3$. In the sentence preceding the theorem we pointed out that the result follows from the classical Lewy extendibility of CR functions (as described in Theorem 10.3) combined with the work of Rossi [R]. The really new work in [HL] and its sequel [HL2] are the global results characterizing boundaries of varieties without mention of the Levi form.

Theorem 10.4. was intended to assert the existence of a variety with smooth boundary and a finite number of isolated interior singularities holomorphically **immersed** into C^n. In the statement, the word "immersed" was erroneously omitted. Its intention is implicit in a serious reading of the result and the material prior to it. (For instance see Example 9.1, the sentence prior to Theorem 9.2, and the first paragraph of Theorem 10.3.) However, to completely clarify Theorem 10.4 we shall correct the wording of the result and then in the Lemma below we shall explicitly establish an equivalent formulation in terms of immersions.

The subsequent applications of Theorem 10.4 appear in two papers: [Y] and [L]. In fact [Y] presents an alternate proof of the Theorem 10.4 which overlooks the possibility of immersions shown in the example above. Curiously, no reference to this appears in [LY]. Nevertheless, as we shall show below, the results in [Y] and the arguments in [L] are easily amended.

To correct the error in exposition in Theorem 10.4 we recall some elementary facts. Let V be a variety with $d[V] = [M]$ as in the main theorem 8.1 of [HL]. Fix $p \in M$ and suppose that in a neighborhood $\mathcal{U}$ of p there is a local component W of V which is a C^k-submanifold with boundary M. Then $\overline{V - W}$ is an analytic subvariety of $\mathcal{U}$ and therefore has a finite number of irreducible components at p. (See [K] or [H].)

Definition. Suppose now that every point $p \in M$ has the property above (as is the case when M is strictly pseudoconvex). Then a point $p \in \overline{V}$ is defined to be an **intrinsic** singular point if it is a singular point of some local irreducible component of V at p if $p \in V$, or of $\overline{V - W}$ if $p \in M$.

Theorem 10.4 should be amended in line 4 by replacing the word "isolated" with the word "intrinsic".

Theorem 10.4. (amended): *Let $M \subset C^n$ be a connected C^k manifold satisfying the hypothesis of* Theorem 8.1*, and suppose M is pseudoconvex. Then there exists an irreducible, p-dimensional complex analytic subvariety $V \subset C^n \backslash M$ with $\overline{V}$ having at most finitely many intrinsic singularities, such that $[M] = d[V]$, with C^k boundary regularity for each local component of V near M.*

The proof should be amended in line 4 to read:

Theorem 10.3a now shows that the *intrinsic singularities of $\overline{V}$ form a compact subvariety of* C^n which must have dimension 0.

As mentioned above the example proclaimed in the title of [LY] appears explicitly in [HL] in Example 9.1. It is the simplest holomorphic immersion $C^2 \to C^3$ with self-intersections. In fact the example F in [LY] differs from Example 9.1 in [HL] by a **linear change of variables**. More precisely, if we define $L(x, y, z) = (4x + 1, z, 8y - 4x)$ and $\lambda(t, z) = (\frac{1}{2}(t + 1), z)$, then $\Phi = L \circ F \circ \lambda$ is exactly Example 9.1.

Example 9.1 [HL] considers the variety V given by the F-image of a ball whose radius r_0 is chosen to be the first r for which the image has a self intersection. This value of r was considered particularly interesting because the boundary M of V is a (strictly pseudoconvex) real analytic submanifold of C^3 and V is a complex submanifold of $C^3 - M$ but the pair is not a topological submanifold-with-boundary. The apparent content of [LY] is to mention that one can also consider $r > r_0$ in this example.

Incidentally, the theorem announced without proof in [LY] follows immediately from [HL]. Luk-Yau assume the additional hypothesis that M is contained in the boundary of a bounded strictly pseudoconvex domain D in C^N. In a neighborhood of M, the subvariety V obtained from Theorem 10.4 has a component W (a "strip") which is a smooth submanifold with boundary $M - N$ where N is a nearby "parallel" manifold. Thus, $V - W$ has boundary N. Since N is contained in a smaller strictly pseudoconvex domain $D(\epsilon) \subset\subset D$, the Stein manifold version of the main result (Theorem 8.6) in [HL], gives a subvariety Z of $D(\epsilon) - N$ with $d[Z] = [N]$. By uniqueness $V - W$ and Z must agree. Hence $V - W$ misses a neighborhood of M. This rules out singular points of V near M. Hence the entire singular subvariety of V reduces to a finite set.

As noted above, the authors of [LY] neglected to mention that the example they present contradicts a result of their own, namely [Y; Thm 5.14 (Thm C in the introduction)]. In proving Theorem 5.12 in [Y], from which 5.14 is stated to be an "easy consequence", Yau constructs a normal variety over C^N, and he carefully points out (on page 89) that self- intersections may occur after projecting to C^N. However, this point is completely ignored in the statement of Theorem 5.14 which should be amended to read: "Then M is the boundary of an *immersed* complex submanifold ...".

To prove this amended statement we use the following.

Theorem 10.4′. *Suppose $M \subset C^n$ is a compact, connected, oriented, maximally complex submanifold of class C^k and dimension $2p - 1 > 1$. Assume M is strictly pseudoconvex. Let $V \subset C^n - M$ be the analytic subvariety of dimension p and of finite volume with $d[V] = [M]$ given by [HL, Thm. 8.1]. Then there exists:*

(i) A compact space $\overline{X} = X \cup \partial X$ with where X is a normal Stein variety having at most a finite number of singular points, and such that $(X, \partial X)$ is a C^k-manifold-with-boundary away from the singular points, and

(ii) A map $\rho : \overline{X} \to C^n$, which is holomorphic on X and of class C^k up to the boundary, inducing a C^k-diffeomorphism from ∂X to M and having $\rho(\overline{X}) = \overline{V}$.

Furthermore, ρ is an immersion outside a finite subset of X which contains the singularities of X and is contained in the preimage of the intrinsic singularities of $\overline{V}$. Finally, when V is a hypersurface, ρ is a local holomorphic embedding.

Proof. Let $\rho_0 : \widetilde{V} \to V$ be the normalization of V. Since V has a finite number of intrinsic singularities, the singular set of $\widetilde{V}$ is finite. We complete $\widetilde{V}$ to $\overline{X}$ as follows. Each $p \in M$ has a neighborhood $\mathcal{U}$ such that $\overline{V} \cap \mathcal{U} = W \cup V_1 \cup \cdots \cup V_m$ where W is a C^k-submanifold with boundary and where $V_1, ..., V_m$ are irreducible subvarieties of $\mathcal{U}$ each of which has a finite singular set (again because the intrinsic singularities are finite). Let $\rho_j : \widetilde{V_j} \to V_j$ be the normalization of V_j. Note that $\widetilde{V_j}$ has a finite singular set and ρ_j is a holomorphic homeomorphism. These maps induce a map

$$\rho_{\mathcal{U}} : W \amalg \widetilde{V_1} \amalg \cdots \amalg \widetilde{V_m} \longrightarrow W \cup V_1 \cup \cdots \cup V_m = \overline{V} \cap \mathcal{U},$$

which is canonically isomorphic to ρ_0 on the preimage of $V \cap \mathcal{U}$ by the uniqueness of normalization. Gluing these pieces to $\widetilde{V}$ and adding the boundary in the obvious way produces X.

Note that X contains no compact subvarieties of positive dimension since ρ has discrete fibres, but would be constant on connected components of such subvarieties. Since ∂X is strictly pseudoconvex we therefore conclude that X is a Stein space by [G].

The last statement is a consequence of the fact that isolated hypersurface singularities are normal.

When $p = n - 1 \geq 3$ the arguments in [Y] apply to show that X is non-singular if and only if the Kohn-Rossi cohomology groups of the boundary complex are 0. This gives the amendment to [Y] discussed above.

Since X is Stein, the arguments on page 13 of [L] which use [HL; 10.4] carry through unchanged.

References

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